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LINEAR VERSUS NONLINEAR MACROECONOMIES: A STATISTICAL TEST

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A statistical test based on the estimated bispectrum is presented which can distinguish between the linear stochastic dynamics widely used in macroeconomic models and alternative nonlinear dynamic mechanisms, including both nonlinear stochastic models and nonlinear deterministic (chaotic) models. The test is applied to an aggregate stock market index and to an aggregate industrial production index. In both cases the test easily rejects the null hypothesis of a linear stochastic generating mechanism. This result strongly suggests that nonlinear dynamics (deterministic or stochastic) should be an important feature of any empirically plausible macroeconomic model.

1. INTRODUCTION

It has recently become possible, by using the estimated bispectrum, to formally test the null hypothesis that the generating mechanism of a macroeconomic time series is linear. Such testing is valuable for two reasons.

First of all, most macroeconomic models are linear or loglinear. Rational expectations models, for example, routinely assume that the stochastic process driving the system is a (linear) VAR process and proceed to generate output series which are low order AR processes. If we can reject the null hypothesis of a linear generating mechanism for typical macroeconomic time series, then these models are misspecified, perhaps seriously.

Why might such misspecification be important? Suppose that a variable $x(t)$ is generated by the nonlinear mechanism

$$x(t) = \beta u(t-1)x(t-2) + u(t) \tag{1.1}$$

where $u(t) \sim$ iid $(0, \sigma^2)$. It can be shown (Granger 1980; Granger and Andersen 1978) that the $x(t)$ so generated are serially uncorrelated. Yet $x(t)$ is clearly forecastable from its own past using an estimated version of equation (1.1). In fact, this model can have an R^2 of up to .50, so the variance of the errors made by the optimal forecast can be as little as half as large as the variance of the errors made by the optimal *linear* forecast. Thus, the assumption that economic agents form expectations based on linear projections may be a very poor representation of rational behavior if the process being forecast is generated by a nonlinear generating mechanism.

Secondly, it has recently been suggested (Benhabib and Day 1980, 1981, 1982;

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Stutzer 1980; and Grandmont 1985) that Samuelson-Gale overlapping generations macroeconomic models need not be driven by exogenous stochastic variables at all. In these authors' work, highly nonlinear deterministic recursions embodying "chaotic" dynamics generate output series which appear to be stochastic. In particular, Grandmont has developed a deterministic macroeconomic model of this type which leads to substantially different (more activist) policy conclusions than do typical rational expectations models. Brock and Chamberlain (1984) have shown that linear spectral methods are unable to distinguish between the output of a nonlinear chaotic model and the output of a Lucas-type rational expectations model. While the bispectral test described below does not directly test for chaotic dynamics *per se*, it can distinguish between the outputs of these two models by testing for the nonlinearity inherent in recursions exhibiting chaotic dynamics.

Thus, a rejection of the null hypothesis of linearity strongly suggests that models such as Benhabib and Day's, Stutzer's, and Grandmont's deserve serious consideration. Such a rejection would also motivate the application of tests capable of distinguishing nonlinear stochastic dynamics from deterministic chaotic dynamics. For example, Brock (1986), Barnett and Chen (1986), and Brock and Sayers (1988) discuss several descriptive statistics which, while not constituting a formal statistical test, may provide insight into whether the hypothesis that a given time series is the result of a deterministic chaotic recursion is, or is not, plausibly consistent with the data. And Brock, Dechert, and Scheinkman (1986) have recently developed a formal statistical test based on the Grassberger-Procaccia-Takens dimension estimate which can distinguish i.i.d. random variates from data generated by a deterministic chaotic recursion.

In our view, however, the important distinction is not between stochastic versus chaotically deterministic dynamics, but rather between linear versus nonlinear dynamics. This notion, first proposed in Ashley and Patterson (1985), is no longer controversial. Indeed, the main thrust of Brock, Dechert, and Scheinkman (1986) is to further develop methods used in testing for chaotic behavior for use in testing for nonlinear dynamics of all sorts. After all, everyone (even Grandmont 1985, p. 1039) recognizes that actual macroeconomic time series are in fact stochastic. Thus, empirically, stochastic modelling is necessary in either case. The crucial issue is: what sort of stochastic dynamics is appropriate? If the underlying dynamics are highly nonlinear, then the standard linear rational expectations macroeconomic models (e.g. Lucas 1972, 1975) are probably quite misleading. In that case, modifying them to be more realistic will probably represent a movement in the direction of the nonlinear chaotic models. Consequently, we focus below on detecting nonlinearity in aggregate economic time series.

The bispectral nonlinearity test mentioned at the beginning of this section is discussed in Section 2. In Sections 3, 4, 5, and 6 below we show that the bispectral test can detect a wide variety of nonlinear generating mechanisms, both stochastic and deterministic. In Sections 7 and 8 we apply the bispectral test to the historical record on two important economic time series: an aggregate asset price series (a widely used index of stock market returns) and an aggregate output series (the index of industrial production in the U.S. manufacturing sector). The null hypothesis of a linear generating mechanism is strongly rejected for both series.

2. A BISPECTRAL TEST FOR DETECTING NONLINEARITY IN TIME SERIES

Subba Rao and Gabr (1980) and Hinich (1982) have both developed statistical tests, based on the estimated bispectrum, for the detection of nonlinearity in a time series. Below we describe and use the Hinich test because it explicitly exploits the asymptotic distribution of the bispectral estimator. Also, Patterson (1983) has developed a computer algorithm implementing the Hinich test and Ashley, Patterson and Hinich (1986) have shown that the test has considerable power to detect stochastic nonlinear dynamics in sample sizes as low as 256.

The bispectrum of a time series is defined as follows. Let $\{x(t)\}$ denote a third-order stationary time series, where t is an integer. To simplify exposition, let $E[x(t)] = 0$. The third order cumulant function is then

$$c_{xxx}(m, n) = E[x(t+n)x(t+m)x(t)].$$

The bispectrum at frequency pair (f_1, f_2) is the double Fourier transform of $c_{xxx}(m, n)$:

$$B_x(f_1, f_2) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{xxx}(m, n) \exp\left[-i2\pi(f_1 m + f_2 n)\right]. \tag{2.1}$$

It is a spatially periodic complex function whose principal domain is the triangular set $\Omega = \{0 < f_1 < 1/2, f_2 < f_1, 2f_1 + f_2 < 1\}$. A rigorous treatment of the bispectrum is given by Brillinger and Rosenblatt (1967).

Since a complete description of the Hinich nonlinearity test based on the estimated bispectrum can be found in Hinich (1982), Hinich and Patterson (1985) and Ashley, Patterson and Hinich (1986), a detailed exposition of the implementation of the test will not be presented here; a condensed exposition can be found in the Appendix. For present purposes it suffices to note that the standardized bispectrum $(2\,\Psi^2(f_1, f_2)$, defined in equation A.6) is a constant for all frequency pairs (f_1 and f_2) if the generating mechanism of $\{x(t)\}$ is linear.

3. ARTIFICIAL DATA: I. STOCHASTIC HICKS ECONOMY

The bispectral test is only useful if it has significant power to detect the kinds of nonlinear generating mechanisms one might expect to find in economic models.

Our first example is data generated from a modification of a much-studied model in the business cycle literature due originally to Hicks (1950). Hicks' theory, in brief, is an elaboration of the Samuelson multiplier-accelerator theory to include a floor level of investment (I_f) and a ceiling level of output (Y_c).

Blatt (1978) has shown that data generated by such a nonlinear model can be analyzed using the usual (linear) econometric methods, yielding results which give no hint of the underlying nonlinearity of the process generating the data. Blatt's model is deterministic (as was Hicks') and yields precisely periodic cycles. We modified his model to eliminate this counterfactual feature by adding a small stochastic error term to the investment equation, yielding:

$$C(t) = a + m[Y(t-1)]$$

$$I(t) = \max\{I_f, I_{eq} + v[Y(t-1) - Y(t-2)]\} + v(t)$$

$$Y(t) = \min\{Y_c, C(t) + I(t)\} \qquad \text{with}$$

$$v(t) \sim NIID[0, (.1)^2].$$

We generated observations on $Y(t)$ using the same parameter values as Blatt—i.e. $v = 2$, $Y_c = 4.0$, $a = .2$, $m = .86$, $I_f = 0$, and $I_{eq} = .3$. A typical realization of this process with 512 observations yields the estimated standardized bispectrum plotted in Figure 1 over the triangular principal domain defined in the Appendix. For clarity's sake, all values of the estimated bispectrum outside this principal domain are set to zero in the plot. This was done because—due to the symmetry inherent in $B_x(f_1, f_2)$ as defined in equation A.1—the points outside the principal domain merely repeat the information within the principal domain. A few of the contours spill over a bit beyond the principal domain nevertheless; this is due to the smoothing algorithm used by the plotter.

The estimated standardized bispectrum of the observations on $Y(t)$ is clearly not flat within the triangular principal domain. Is the estimated standardized bispectrum significantly nonflat? The nonlinearity test statistic for this realization is $Z = 4.30$. (Z is defined in equation A.15 in the Appendix.) Since $Z \sim N(0, 1)$ under the null hypothesis of a linear generating mechanism, we can reject the null hypothesis at the .001 percent level.

We conclude that the bispectral test can detect the kinds of nonlinearity which capacity and nonnegativity constraints induce.

4. ARTIFICIAL DATA: II. NONLINEAR DETERMINISTIC DYNAMICS

As noted in the introduction, a good deal of attention is currently focused on a particular class of deterministic nonlinear generating mechanisms yielding so-called "chaotic" dynamics. In this section we examine the ability of the bispectral test to detect the kinds of nonlinearities inherent in such dynamics.

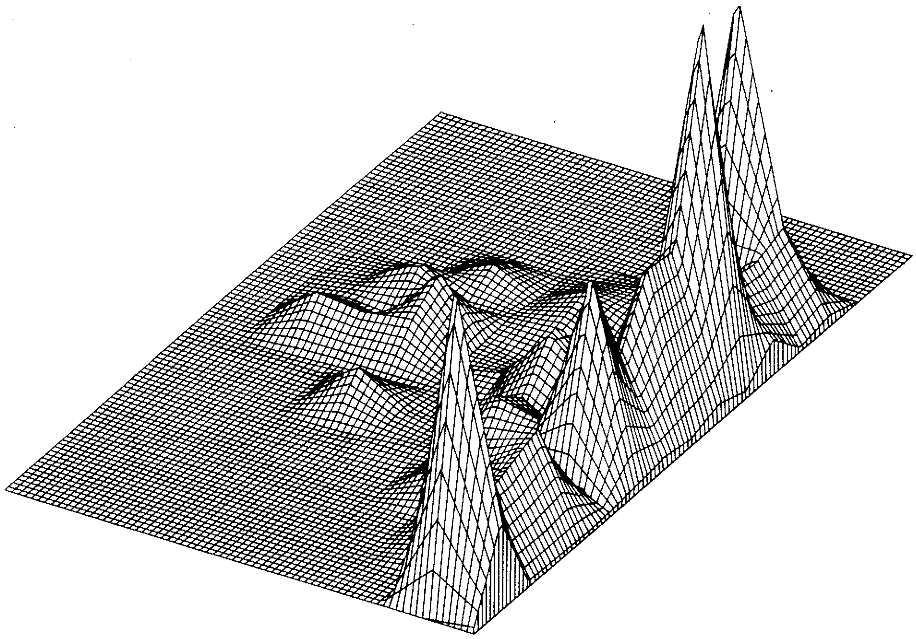
Chaotic dynamical systems are characterized by deterministic recurrence relations of the form

$$x(t+1) = G(x(t))$$

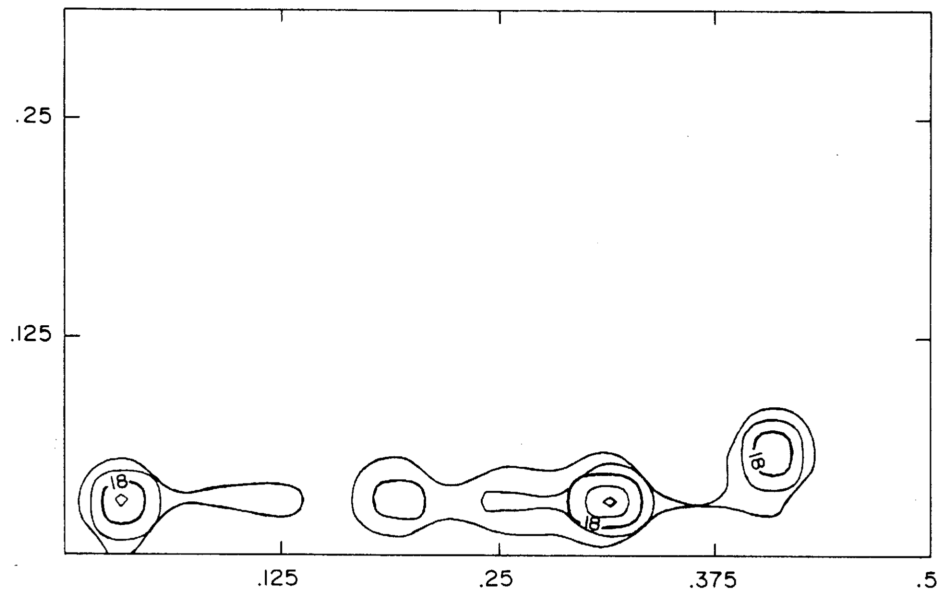
such that the sequence $\{x(t)\}$ tends toward neither an equilibrium point nor a limit cycle. (Technically, the system has a "strange attractor." See Ruelle and Takens 1971; Li and Yorke 1975; Brock 1986; Barnett and Chen 1986; and Ramsey and Yuan 1987, for further details.) Bunow and Weiss (1979) list four chaotic recursions which have received explicit attention in the literature.[2]

1. Logistic: $x_{t+1} = rx_t(1 - x_t) \qquad 0 < r \leq 4.$
2. Exponential Logistic: $x_{t+1} = x_t \exp[r(1 - x_t)] \qquad r > 0.$

[2] See Pohjola (1981) for an application of the logistic recursion to a Goodwin-type growth cycle model.



A) BAS-RELIEF MAP



B) CONTOUR PLOT

FIGURE 1

ESTIMATED STANDARDIZED BISPECTRUM FOR STOCHASTIC HICKS MODEL

3. Triangular: $x_{t+1} = 1 - 2|x_t - 1/2|$.
4. Cubic: $x_{t+1} = x_t + ax_t(x_t^2 - 1)$.

Example time series, generated from the logistic and exponential logistic recursions were plotted for a variety of values of the parameter r. These series did not resemble typical economic data in that certain values (ca. 0 and 1 for the logistic recursion, ca. 0 for the exponential recursion) were substantially more prevalent than all other values. A typical series for the logistic recursion with $r = 4$ yielded the estimated standardized bispectrum plotted in Figure 2. As was typically the case with the logistic-type series, the estimated standardized bispectrum was small with an occasional very large value. The test statistic Z (defined in equation (A.15)) is ill-suited for detecting such spikes because the 80 percent quantile ignores the largest 20 percent of the sample of standardized bispectral estimates, but the non-constancy of the standardized bispectrum in these cases is obvious by inspection. Thus, the estimated bispectrum is likely to detect these models, but they are unlikely candidates for economic modelling since their output so little resembles typical economic data.[3]

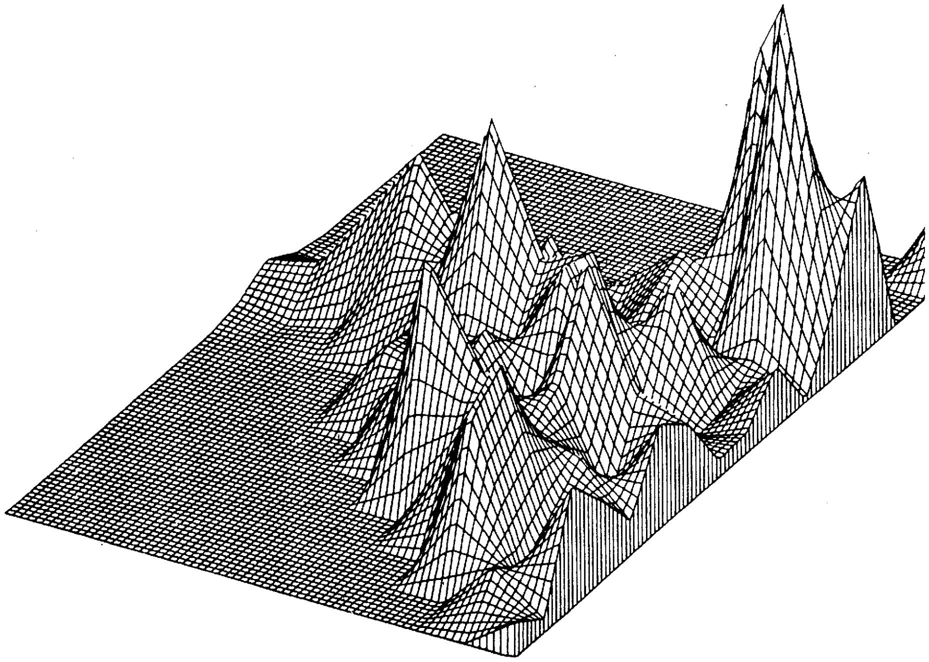
In contrast, the triangular recursion yields time series which look quite like many stationary economic time series. (See Figure 3 for an example.) The recursion output is uniformly distributed, whereas most economic data looks more gaussian, but at least its distribution is smooth. The estimated standardized bispectrum for this series is plotted in Figure 4. It obviously has significant dispersion over the frequency pairs. Using a sample size of 456 (typical of the monthly postwar record) this dispersion is reflected in a Z statistic of 3.35. Z is a unit normal under the null hypothesis of a linear generating mechanism, hence the null hypothesis is rejected at the .04 percent level.[4]

The cubic recursion looks like economic data only for values of a close to 4. As it stands, our test is not useful in detecting this form of nonlinearity, however. In fact, our simulations show that the estimated bispectrum of data generated from this model converges to zero at all frequency pairs as the sample size increases. Evidently, the population third-order cumulants are zero in this case due to the particular symmetry of the model. We conjecture that a test based on this property or on a higher-order estimated polyspectrum would detect this form of chaotic dynamics, but such a test might well require more data than is available in the postwar monthly record.

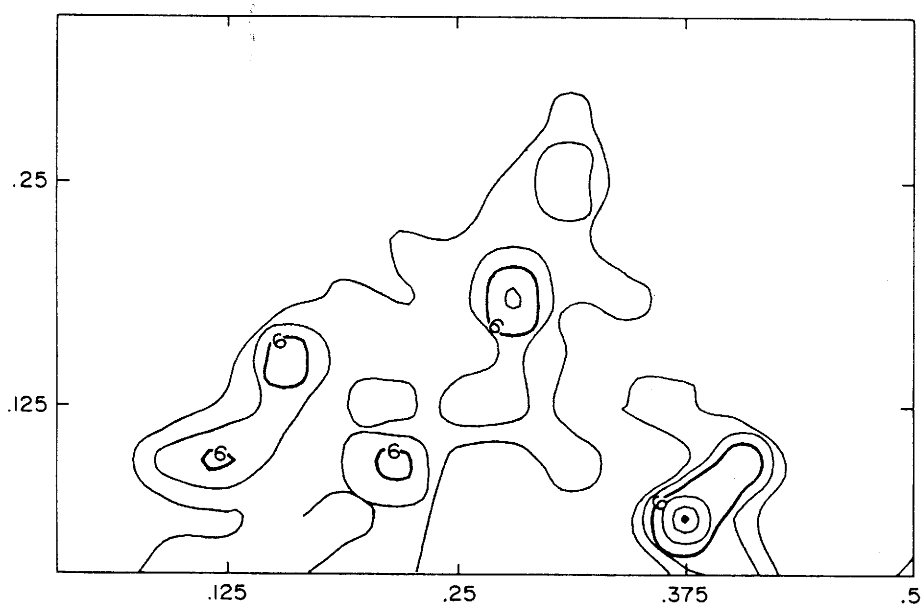
We conclude that the estimated bispectrum is capable of detecting many (albeit not all) of the simple forms of chaotic dynamics which have been considered in the mathematics literature. Chaotic behavior has also been shown to arise in more complex models which are of explicit economic interest. In particular, in the next

[3] In addition, we found that minor rounding errors in the computations of these logistic-type recursions caused a prompt decay to an equilibrium point at zero or one. We eliminated this problem by using integer arithmetic, but we note that the knife-edge behavior of these recursions makes it unlikely that they could ever be observed in economic processes.

[4] The triangular recursion yields trajectories which appear to be white noise. Sakai and Tokumaru (1980) show that simple variations on this recursion yield trajectories which are apparently AR(1) processes.



A) BAS-RELIEF MAP



B) CONTOUR PLOT

FIGURE 2

ESTIMATED STANDARDIZED BISPECTRUM FOR LOGISTIC MODEL

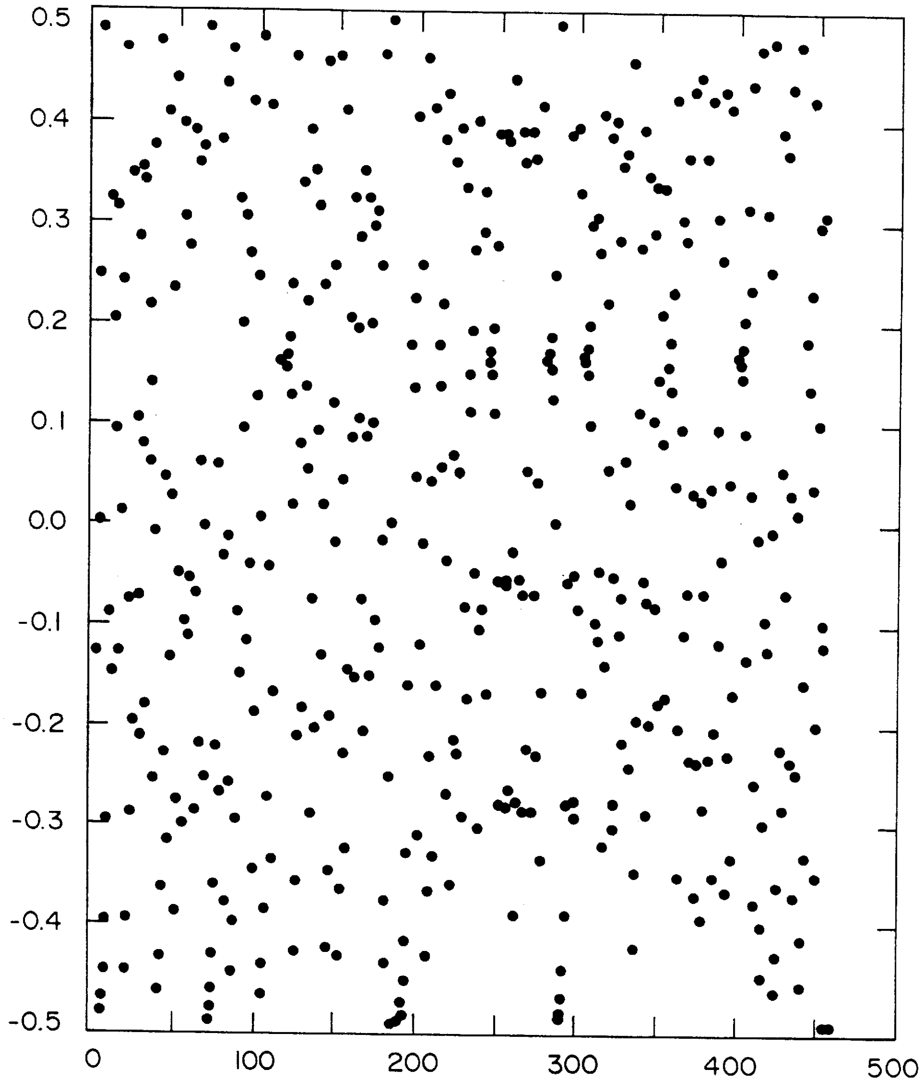


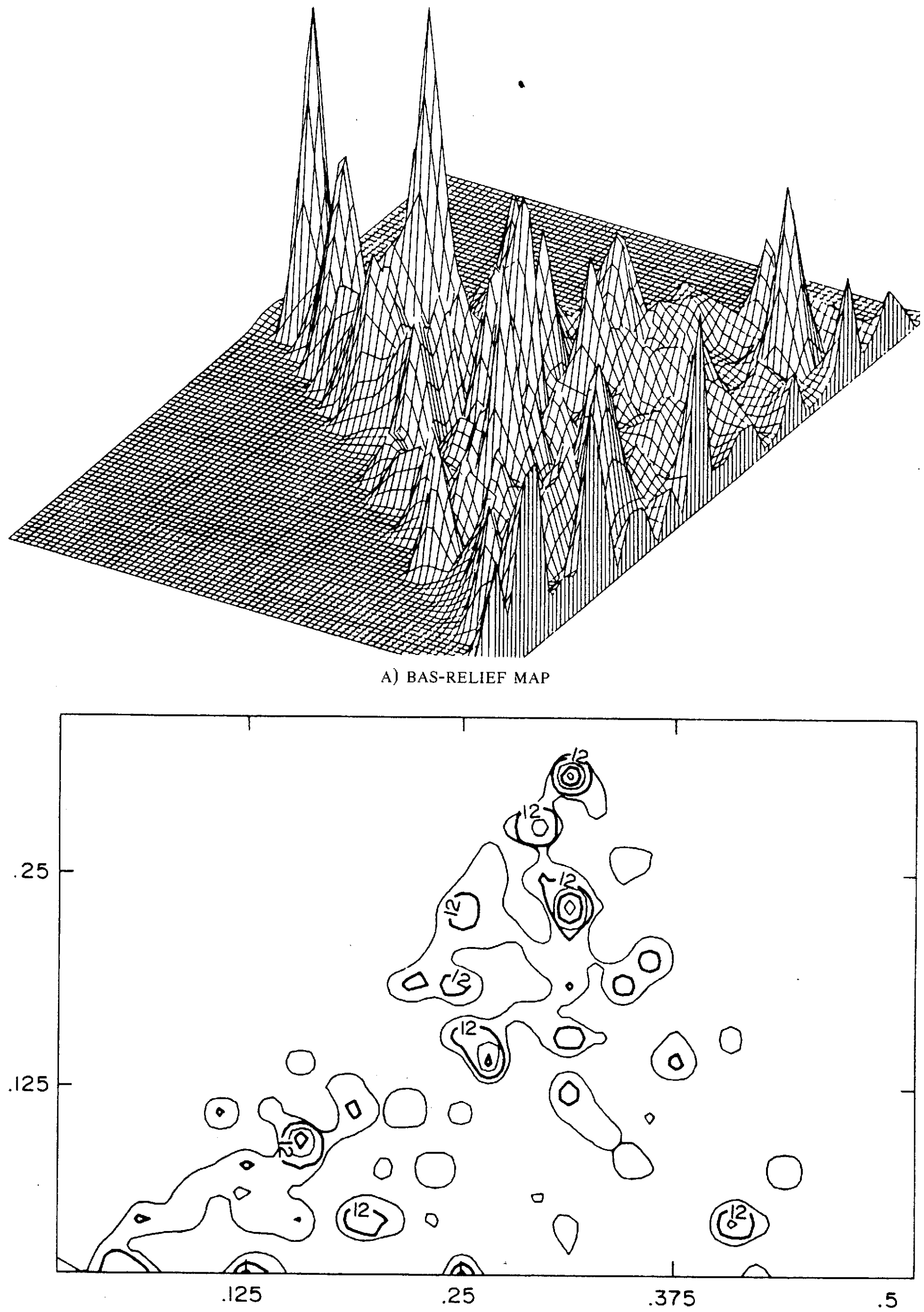
FIGURE 3

PLOT OF TRIANGLE RECURSION

section we examine the ability of the bispectral test to detect nonlinear dynamics in an ISLM macro model.

5. ARTIFICIAL DATA: III. A NONLINEAR KEYNESIAN MODEL

Day and Shafer (1985) have shown that chaotic behavior can easily arise in the standard, fixed price Keynesian macroeconomic model with a nonlinear investment schedule.



A) BAS-RELIEF MAP

B) CONTOUR PLOT

FIGURE 4

ESTIMATED STANDARDIZED BISPECTRUM FOR TRIANGULAR MODEL

A specific example of one of the models they consider is:

GNP Identity: $Y(t+1) = \text{Min}\,\{C[Y(t)] + I[r(t)] + A,\ Y^{\text{capacity}}\}$.

Consumption Function: $C[Y(t)] = \beta Y(t)$ with $0 < \beta < 1$.

Investment Function: $I[r(t)] = \begin{cases} \mu[r'' - r(t)]/r(t) & 0 \leqslant r(t) \leqslant r''. \\ 0 & r(t) > r'' \end{cases}$

Money Demand Function: $M(t) = kY(t) + \lambda/[r(t) - r^*]$ $\quad r(t) \geqslant r^* \geqslant 0$.

This money demand function implies an LM curve with the usual shape. Note that the investment schedule is downward sloping in the interest rate with a floor of zero. In short, this model is consistent with the usual textbook presentation.

Day and Shafer give a sufficient condition on the parameters of this model for the sequence $\{Y(t)\}$ to be chaotic.[5] The parameter choices $Y^{\text{capacity}} = 100$, $A = 13$, $\beta = .7$, $\mu = 500$, $r'' = .25$, $r^* = .01$, $M = 40$, $k = .2$, and $\lambda = 4.8$ satisfy this sufficient condition. These parameters lead to steady state values of 96.7 for Y, 67.7 for C, 16 for I, and .24 for r. These are reasonable for a high output, high interest rate economy.

We simulated this model over 456 periods, the length of the monthly post war record. The first 100 values of the resulting $Y(t)$ sequence are plotted in Figure 5. No particular claim as to realism is made here for this sequence (actual GNP is trended and much smoother); none is necessary in any case since this model is obviously highly stylized. However, the sequence does appear to be stationary non-gaussian noise. The correlogram and partial correlogram look quite ordinary; this series would no doubt be modelled as an AR(4).

Applying the bispectral test to these data yields the estimated standardized bispectrum plotted in Figure 6. The resulting test statistic is $Z = 2.39$, so that the null hypothesis of a linear generating mechanism can be rejected at the 1 percent level. Thus, the (chaotic) nonlinear dynamics in this deterministic Keynesian model are easily detected by the bispectral test.

6. ARTIFICIAL DATA: IV. NONLINEAR STOCHASTIC DYNAMICS

In Section 3 we examined the ability of the bispectral test to detect a specific type of nonlinear stochastic generating mechanism of particular economic interest. Time series analysts have explicitly considered a number of additional nonlinear stochastic generating mechanisms. While these mechanisms have arisen in noneconomic contexts, they may be important in economic modelling as well.

Ashley, Patterson and Hinich (1986) have used simulation methods to estimate the power of the bispectral test for detecting many of these mechanisms. Here we present a very brief summary of those results.

[5] This is Proposition 3 in their paper. Note that a needed minus sign is missing from the right hand side of equation 9.

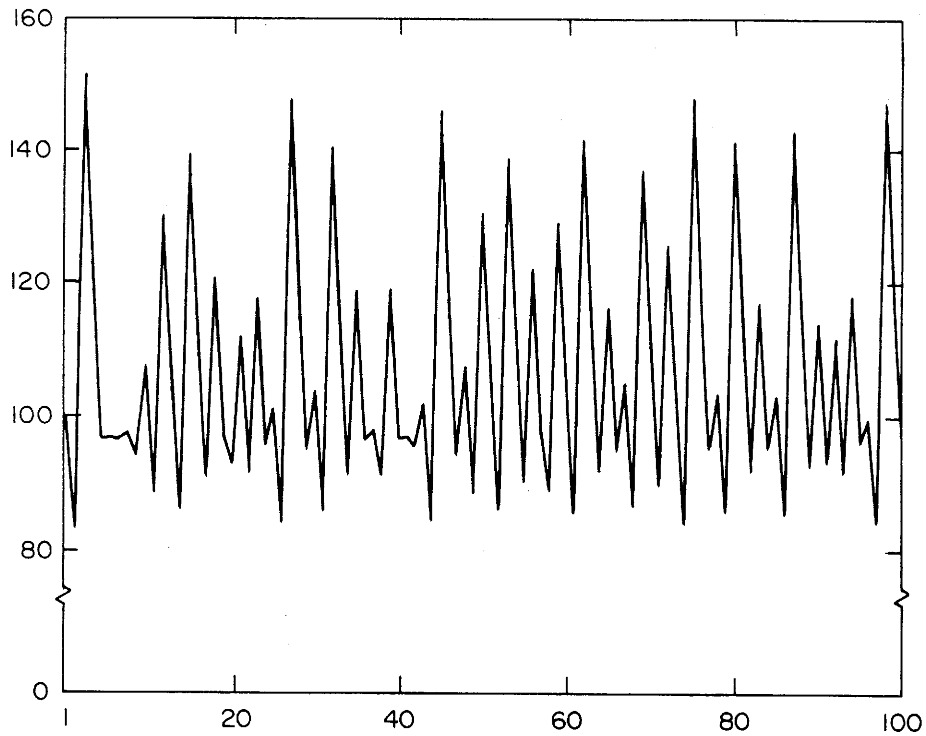


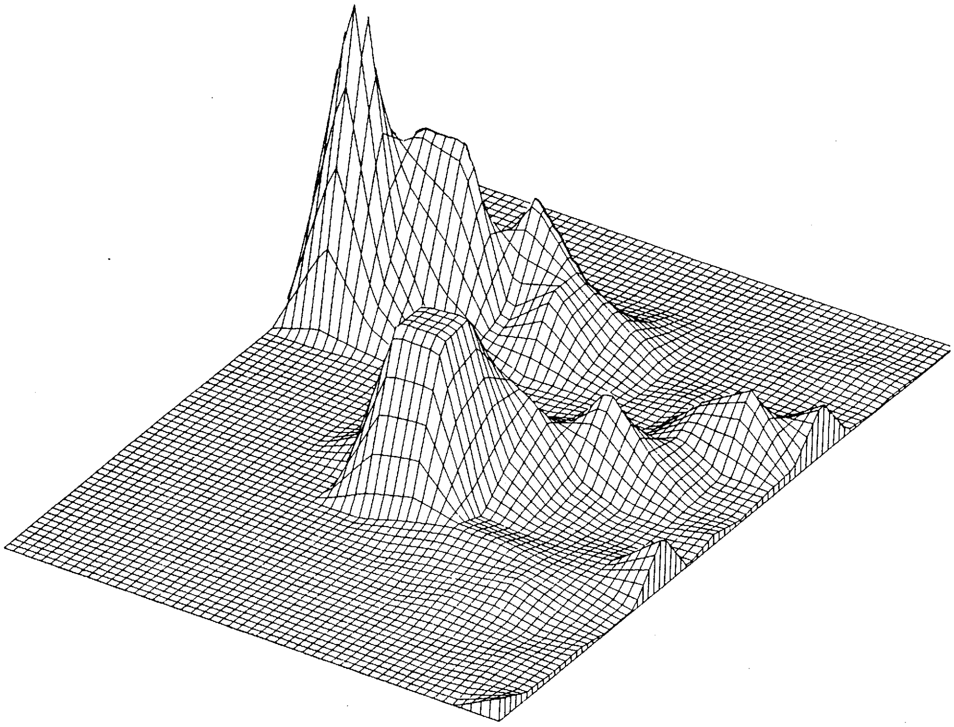
FIGURE 5

NONLINEAR KEYNESIAN MODEL—GNP TIME PATH

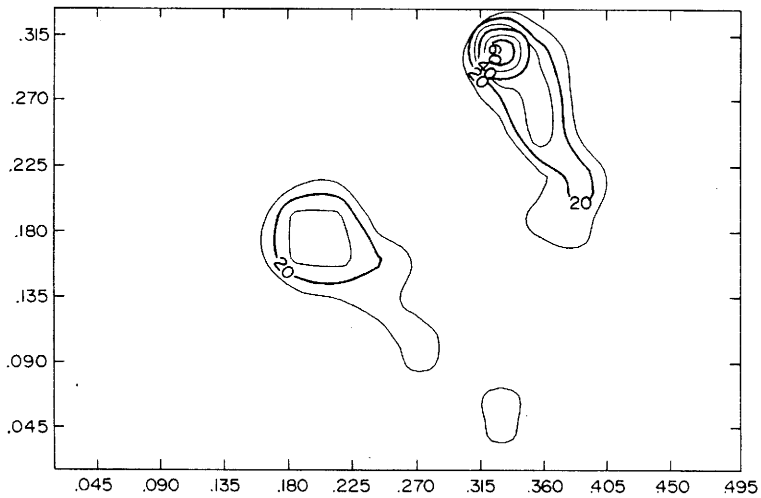
Letting $\varepsilon\{t\} \sim NIID(0, \sigma^2)$, the particular models they considered are:

a. Bilinear: $x(t) = \varepsilon(t) + .7\varepsilon(t-1)x(t-2)$.
b. Nonlinear MA: $x(t) = \varepsilon(t) + .8\varepsilon(t-1)\varepsilon(t-2)$.
c. Extended Nonlinear MA: $x(t) = \varepsilon(t) + .8\varepsilon(t-1)\sum_{j=2}^{20}(.8)^{j-2}\varepsilon(t-j)$.
d. Nonlinear AR: $x(t) = [.5 + .5\varepsilon(t-1)]x(t-1) + \varepsilon(t)$.
e. Threshold AR: $x(t) = -.5x(t-2) + \varepsilon(t)$ if $x(t-1) \leqslant 1$, $x(t) = .4x(t-1) + \varepsilon(t)$ otherwise.
f. Nonlinear Threshold AR: $x(t) = -(.1 + .4|x(t-1)|)x(t-1) + \varepsilon(t)$ if $|x(t-1)| \leqslant 1$, $x(t) = -.5x(t-1) + \varepsilon(t)$ otherwise.
g. Exponential AR: $x(t) = (.9 + .1 \exp[-x^2(t-1)])x(t-1) - (.2 + .1 \exp[-x^2(t-1)])x(t-2) + \varepsilon(t)$.

Ashley, Patterson and Hinich give explicit references to the literature for each of these models. Table 1 summarizes their results on the power of the bispectral test, averaged over 250 simulations, for data generated from each of the eight models. These results show that the bispectral test has substantial power to detect many of the kinds of nonlinear generating mechanisms which have appeared in the time series literature.



A) BAS-RELIEF MAP



B) CONTOUR PLOT

FIGURE 6

ESTIMATED STANDARDIZED BISPECTRUM FOR KEYNESIAN MODEL

TABLE 1
POWER OF 5 PERCENT BISPECTRAL TEST

Model	Sample Size		
	256	512	1024
a. Bilinear	.78	.96	1.00
b. NL MA	.54	.75	.96
c. Extended NL MA	.71	.92	1.00
d. NL AR	.76	.95	1.00
e. Threshold AR	.33	.55	.80
f. NL Threshold AR	.01	.01	.00
g. Exponential AR	.03	.00	.00

7. HISTORICAL DATA: I. COMMON STOCK RETURNS

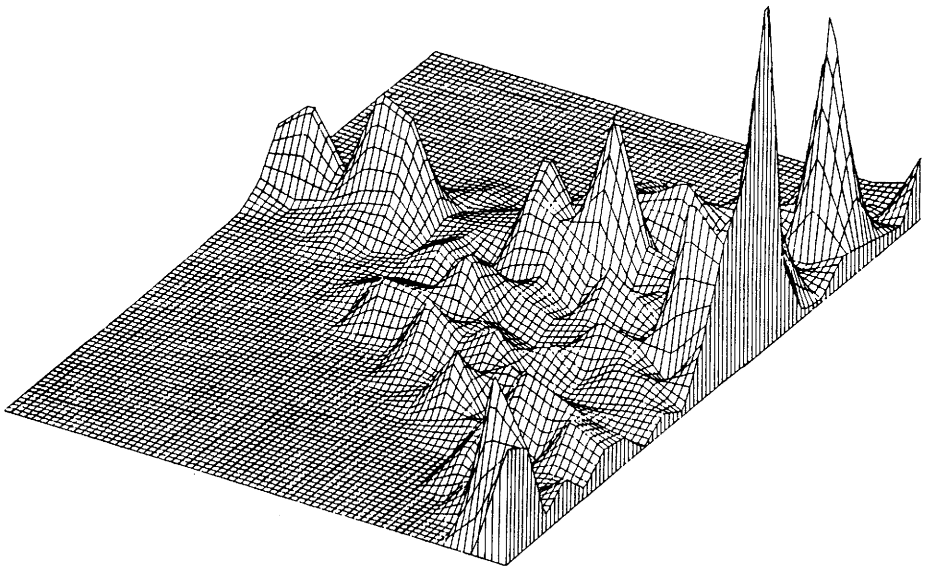
We tested two economic time series for nonlinearity using the bispectral test. The first time series is the daily percentage change in the CRSP value-weighted common stock index. (CRSP denotes the Center for Research in Security Prices at the University of Chicago.) This index includes all common stocks listed on the New York or American Stock Exchange. Using a sample of 1,000 trading days from January 20, 1981 to December 31, 1984, the estimated standardized bispectrum is plotted in Figure 7. The resulting test statistic is $Z = 4.02$, so that the null hypothesis of a linear generating mechanism for this time series can be rejected at the .003 percent level.[6]

8. HISTORICAL DATA: II. INDEX OF INDUSTRIAL PRODUCTION

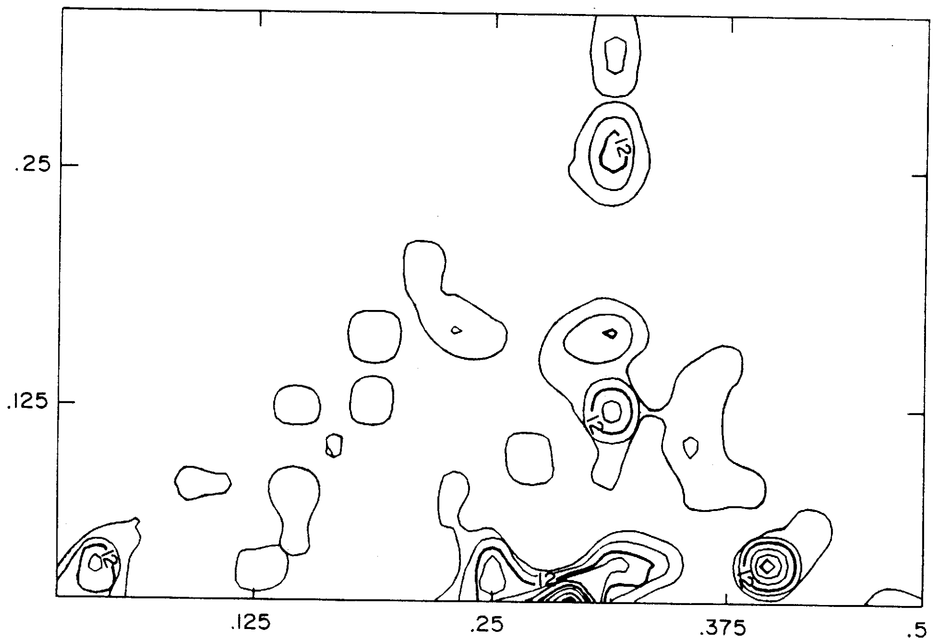
The second economic time series analyzed is the monthly growth rate in the index of industrial production for the U.S. manufacturing sector as reported by the Federal Reserve Board. We first analyzed this time series in seasonally adjusted form. Using 456 observations from February 1947 to January 1985, the bispectral test yielded $Z = 4.58$, which is significant at the .002 percent level. We then analyzed this series in seasonally unadjusted form, to rule out the possibility that the test was detecting some nonlinearity in the seasonal adjustment process. The behavior of the unadjusted series is dominated by a deterministic calendar effect—e.g., the growth rate is substantially below the mean every July. (As mentioned in Section 2, the theory underlying the bispectral test assumes that the time series is stationary. The effect of the nonstationarity in this case was to induce absurdly large peaks in the estimated bispectrum at certain seasonal frequency pairs.) We corrected for this calendar effect by regressing the series against 11 monthly dummy variables.[7] The estimated standardized bispectrum of the resulting

[6] Related evidence of serial dependence (i.e. non i.i.d.ness) in stock returns can be found in Ashley and Patterson (1986) and Scheinkman and Le Baron (1986).

[7] Since the sample size is large, it is reasonable to neglect the sampling variance in the estimated coefficients in this regression. Note that the purpose of this (linear) deseasonalizing filter is to remove a deterministic nonstationarity in the series, not to filter out seasonal autocorrelations. The latter is unnecessary since the equivalence theorem proven in Ashley, Hinich, and Patterson (1986, p. 174) shows



A) BAS-RELIEF MAP



B) CONTOUR PLOT

FIGURE 7

ESTIMATED STANDARDIZED BISPECTRUM FOR STOCK MARKET INDEX (apparently stationary) time series is plotted in Figure 8. The bispectral test yielded $Z = 4.85$, which is significant at the .0001 percent level.[8]

9. CONCLUSIONS

We find that the bispectral nonlinearity test resoundingly rejects the null hypothesis of a linear generating mechanism for both the aggregate stock market index and the aggregate industrial production index. We conclude that at least these key macroeconomic aggregates are the output of a nonlinear dynamic system.

This result is consistent with the nonlinear deterministic (chaotic) dynamics proposed by Benhabib and Day, Stutzer, Day and Shafer, and Grandmont. It is also consistent with nonlinear stochastic dynamics, including (but not limited to) bilinear models, stochastic models with inequality constraints, and ARCH in mean models.

This result is not consistent with linear stochastic dynamics. This implies that the linear (loglinear) macroeconomic models usually considered are misspecified. While we cannot at this point quantify how serious this misspecification is, our results are so strong that it is likely that the nonlinearities involved are quite sizeable. If so, then the linear models in common use today may be very misleading.

Finally, from a theoretical point of view, these results also imply that the linear forecasting rules usually invoked to generate ''rational'' expectations in modern macroeconomic models are in fact suboptimal. Consequently, such expectations are not, strictly speaking, rational.

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APPENDIX
THE HINICH BISPECTRAL NONLINEARITY TEST

Let $\{x(t)\}$ denote a third-order stationary time series, where t is an integer. To simplify exposition, let $E[x(t)] = 0$. The third order cumulant function is defined to be

$$c_{xxx}(m, n) = E[x(t + n)x(t + m)x(t)].$$

The bispectrum at frequency pair (f_1, f_2) is the double Fourier transform of $c_{xxx}(m, n)$:

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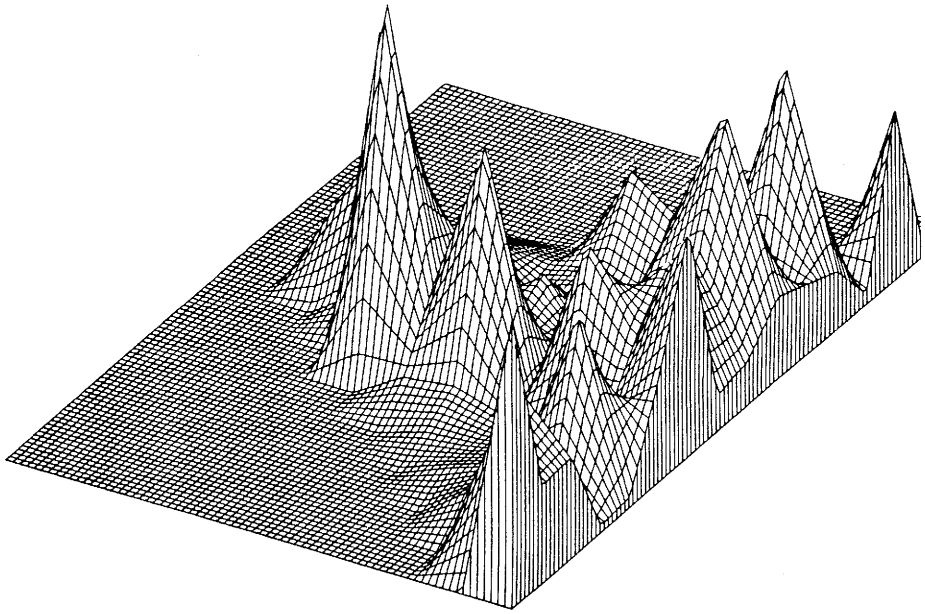
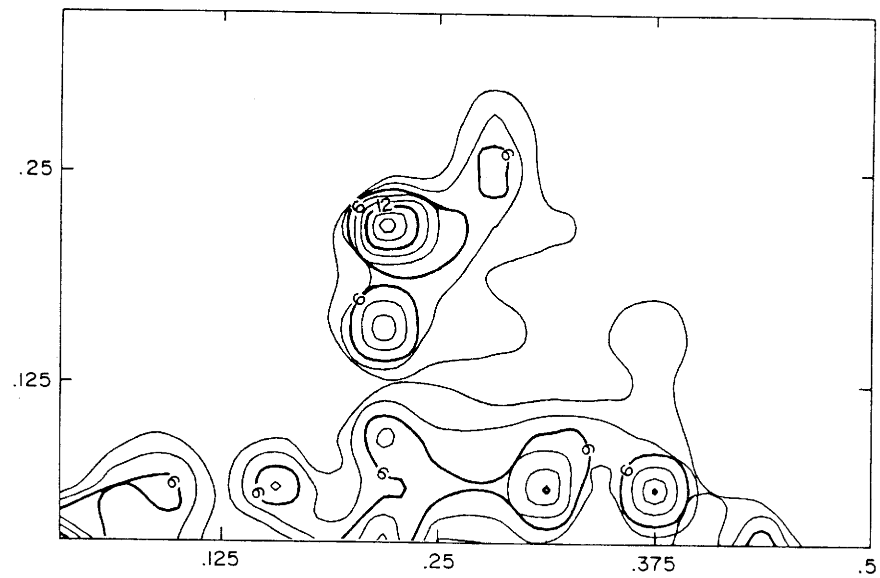
that the standardized bispectrum is invariant to any linear filter. Thus, the only problem associated with sampling errors in the estimated coefficients of the deseasonalizing filter is that they allow small amounts of deterministic (nonstationary) seasonality to remain in the deseasonalized series. The estimated bispectrum of the series deseasonalized in this way shows no evidence of such nonstationarity, however.

[8] Recently, Brock and Sayers (1988) have reported significant rejections of the null hypothesis of linearity applying the Brock-Dechert-Schienkman test to prewhitened deseasonalized monthly data on the index of industrial production for all sectors.



A) BAS-RELIEF MAP



B) CONTOUR PLOT

FIGURE 8

ESTIMATED STANDARDIZED BISPECTRUM FOR INDUSTRIAL PRODUCTION INDEX

(A.1) $$B_x(f_1, f_2) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{xxx}(m, n) \exp[-i2\pi(f_1 m + f_2 n)].$$

It is a spatially periodic complex function whose principal domain is the triangular set $\Omega = \{0 < f_1 < 1/2, f_2 < f_1, 2f_1 + f_2 < 1\}$. A rigorous treatment of the bispectrum is given by Brillinger and Rosenblatt (1967).

Suppose that $\{x(t)\}$ is linear, that is, it can be expressed as

(A.2) $$x(t) = \sum_{n=0}^{\infty} a(n)u(t-n),$$

where $\{u(t)\}$ is purely random (i.e. stationary and serially independent) and the weights $\{a(n)\}$ are fixed. Assuming that $\Sigma_{n=0}^{\infty} |a(n)|$ is finite, the bispectrum of $\{x(t)\}$ is

(A.3) $$B_x(f_1, f_2) = \mu_3 A(f_1) A(f_2) A^*(f_1 + f_2)$$

where $\mu_3 = E[u^3(t)]$,

(A.4) $$A(f) = \sum_{n=0}^{\infty} a(n) \exp(-i2\pi f n),$$

and $A^*(f)$ is its complex conjugate. Since the spectrum of $\{x(t)\}$ is

(A.5) $$S_x(f) = \sigma_u^2 |A(f)|^2,$$

it follows from (A.3) that

(A.6) $$\Psi^2(f_1, f_2) \equiv \frac{|B_x(f_1, f_2)|^2}{S_x(f_1) S_x(f_2) S_x(f_1 + f_2)} = \frac{\mu_3^2}{\sigma_u^6}$$

for all f_1 and f_2 in Ω.

The left hand side of (A.6) defines the square of the skewness function of $\{x(t)\}$, $\Psi(f_1, f_2)$. Thus, the squared skewness function is a constant if $\{x(t)\}$ is linear. This property is the basis for the Hinich linearity test.

The bispectrum of the stationary process $\{x(t)\}$ can be consistently estimated using a sample $\{x(0), x(1), \ldots, x(N-1)\}$ as follows. Let

(A.7) $$F_x(j, k) = X(j/N) X(k/N) X^*((j+k)/N),$$

where j and k are integers and

(A.8) $$X(j/N) = \sum_{t=0}^{N-1} x(t) \exp(-i2\pi jt/N).$$

$X(0)$ is a set to zero; this is equivalent to subtracting off the sample mean of $\{x(t)\}$.

$F_x(j, k)$ is an estimator of the bispectrum of $\{x(t)\}$ at frequency pair (j, k).

However, it must be smoothed—averaged over adjacent frequency pairs—in order to obtain the consistent estimator, $\hat{B}_x(m, n)$[1]

$$\text{(A.9)} \qquad \hat{B}_x(m, n) = M^{-2} \sum_{j=(m-1)M}^{mM-1} \sum_{k=(n-1)M}^{nM-1} F_x(j, k).$$

$\hat{B}_x(m, n)$ is the average value of $F_x(j, k)$ over a square of M^2 points, where the centers of the squares are defined by the lattice

$$L = \{(2m-1)M/2, (2n-1)M/2 \colon m = 1, \ldots, n \quad \text{and} \quad m \leqslant N/2M - n/2 + 3/4\}$$

in the principal domain.

This averaging procedure is precisely analogous to smoothing the periodogram to obtain a consistent estimator of the spectrum. As in that case, smoothing reduces the finite sample variance at the cost of introducing bias. Let

$$\text{(A.10)} \qquad \hat{X}_{m,n} = \frac{\hat{B}_x(m, n)}{[N/M^2]^{1/2}[\hat{S}_x(g_m)\hat{S}_x(g_n)\hat{S}_x(g_{m+n})]^{1/2}}$$

where

$$\text{(A.11)} \qquad g_j = (2j-1)M/(2N)$$

and S_x is the usual (smoothed) estimator of the power spectrum of $\{x(t)\}$. Hinich (1982) shows that the estimators $2|\hat{X}_{m,n}|^2$ are asymptotically distributed as independent, non-central chi-squared variates [i.e. $\chi^2(2, \lambda_{m,n})$] with non-centrality parameter

$$\text{(A.12)} \qquad \lambda_{m,n} = 2(N/M^2)^{-1}|B_x(m, n)|^2/[S_x(m)S_x(n)S_x(m+n)]$$

for all m and n such that the lattice square lies entirely within the principal domain. Below, P denotes the number of such frequency pairs; also, we will refer to $2|\hat{X}_{m,n}|^2$ as the estimated standardized bispectrum.

Under the null hypothesis that $\{x(t)\}$ is linear, equation (A.12) implies that $\lambda_{m,n}$ is a constant independent of m and n. This constant is consistently estimated by

$$\text{(A.13)} \qquad \hat{\lambda}_o = \left\{2 \sum_m \sum_n |\hat{X}_{m,n}|^2/P\right\} - 2.$$

If the null hypothesis is true, then the estimators $2|\hat{X}_{m,n}|^2$ (asymptotically) constitute P independent picks from the $\chi^2(2, \hat{\lambda}_o)$ distribution. They should therefore have a sample dispersion consistent with that distribution. In contrast, if the null hypothesis is false—so that $\{x(t)\}$ is *not* the result of a linear filter applied to i.i.d. noise—then the $\lambda_{m,n}$'s are not all the same. Consequently, the observations on the estimated standardized bispectrum (i.e. on the values of $2|\hat{X}_{m,n}|^2$) are P

[1] Hinich (1982) showed that consistent estimation of $B_x(f_1, f_2)$ requires $N^{.5} < M < N$. However, the Ashley, Patterson and Hinich simulation results showed that the finite-sample size of the nonlinearity test converges to the asymptotic size more quickly for values of $M \simeq .7N^{.5}$.

independent picks from non-central chi-squared distributions with differing non-centrality parameters. Therefore, they should have a sample dispersion exceeding that expected under the null hypothesis of linearity.

This dispersion can be measured in many ways. Based on the simulation results reported in Ashley, Patterson and Hinich, we use the 80 percent quantile of the empirical distribution in the results reported below. This statistic is robust with respect to outliers and its asymptotic sampling distribution is easily calculated. In particular, David (1970), Theorem 9.2 shows that the sample 80 percent quantile, $\hat{\xi}_{.8}$, is asymptotically distributed as $N(\xi_{.8}, \sigma_o^2)$, where σ_o^2 is consistently estimated by

$$\hat{\sigma}_o^2 = .8(1 - .8)f^{-1}(\hat{\xi}_{.8})P^{-1}, \tag{A.14}$$

and $\xi_{.8}$ is the population 80 percent quantile of $\chi^2(2, \lambda_o)$, and $f(\cdot)$ is the density function of $\chi^2(2, \hat{\lambda}_o)$. Thus,

$$Z \equiv \hat{\xi}_{.8}/\hat{\sigma}_o \sim N(0, 1) \tag{A.15}$$

under the null hypothesis that the time series $\{x(t)\}$ is a realization of a linear process as in (A.2).

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